

The Momentum Map, Symplectic Reduction and an Introduction to Brownian Motion

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Abstract

The underlying goal of this Master's thesis is of laying down, in so far as possible, the foundations for later work in Geometric Stochastic Mechanics. The first part is a presentation of symplectic reduction, going through the momentum map and culminating with an explicit construction of a symplectic form on orbits of the coadjoint action of a Lie group. I have made an effort to be as explicit and precise as possible, reviewing many fundamental concepts so that this paper should be readable by anyone who knows the fundamentals of Hamiltonian mechanics as presented, for example, in chapters 5-7 of "Introduction to Mechanics and Symmetry" by Marsden and Ratiu. The second part conveys an introduction to Brownian motion, presenting some of its fundamental properties, defining the Wiener measure and discussing the weak and strong Markov properties.

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1 Lie Groups, Subgroups, Group Actions and Quotients

1.1 Lie groups and subgroups

This section is meant as a review of Lie group theory, presenting some fundamental results that will be needed in this work. It is assumed the reader knows what a Lie group and its Lie algebra are. See [5], chapters 4, 9 and 20 or [6], chapter 4.1 for a detailed exposition of what follows.

Definition 1 *Let G, H be Lie groups. A group homomorphism $F : G \rightarrow H$ that is also a smooth map is called Lie group homomorphism.*

Theorem 2 *Let G and H be Lie groups and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. Suppose $F : G \rightarrow H$ is a Lie group homomorphism. For every $X \in \mathfrak{g}$, there is a unique vector field in $\mathfrak{h}$ that is F -related to X . If we denote this vector field by F_*X , the map $F_* : \mathfrak{g} \rightarrow \mathfrak{h}$ so defined is a Lie algebra homomorphism.*

Definition 3 *A Lie subgroup H of a Lie group G is a Lie group that can be injectively immersed into G . We shall sometimes call the injection itself a Lie subgroup.*

Proposition 4 *Let G be Lie group and suppose $H \subset G$ is a subgroup that is also an embedded submanifold. Then H is a closed Lie subgroup of G .*

Definition 5 *Let G be a Lie group. We define a one-parameter subgroup of G to be a Lie group homomorphism $F : \mathbb{R} \rightarrow G$.*

Theorem 6 *Let G be a Lie group. The one-parameter subgroups of G are precisely the integral curves of left-invariant vector fields starting at the identity. We thus have the following bijection:*

$$\{\text{one-parameter subgroups of } G\} \longleftrightarrow T_e G = \mathfrak{g}.$$

Proof. Let us be given a one-parameter subgroup $F : \mathbb{R} \rightarrow G$. If we define $X := F_* \frac{d}{dt}$ as above and verify that F is an integral curve for X , we will have proven our result by uniqueness of integral curves. However,

$$F'(t_0) = F_* \frac{d}{dt} \Big|_{t=t_0} = X_{F(t_0)}.$$

■

Definition 7 Let G be a Lie group with Lie algebra $\mathfrak{g}$. We define $\exp : \mathfrak{g} \rightarrow G$ by $\exp(X) = F(1)$, where F is the one-parameter subgroup of G generated by X .

Theorem 8 Some Properties of the Exponential Map

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

- (a) The exponential map is smooth,
- (b) for any $X \in \mathfrak{g}$, $F(t) = \exp(tX)$ is the one-parameter subgroup of G generated by X ,
- (c) for any $X \in \mathfrak{g}$, $\exp((s+t)X) = \exp(sX) \exp(tX)$,
- (d) the pushforward $\exp_* : T_0\mathfrak{g} \rightarrow T_eG$ is the identity, under the usual canonical identifications of both spaces with $\mathfrak{g}$,
- (e) for any Lie group homomorphism $F : G \rightarrow H$, $\exp \circ F_* = F \circ \exp$.

Theorem 9 Closed Subgroup Theorem

Suppose G is a Lie group and $H \subset G$ is a subgroup that is also a closed subset of G . Then H is an **embedded** Lie subgroup.

A fundamental and very usefull property that can be evidenced in the proof of the closed subgroup theorem, and that remains true for Lie subgroups in general is the following Lemma.

Lemma 10 Let $i : H \rightarrow G$ be a Lie subgroup of G . Then, for $\xi \in \mathfrak{g}$, we have

$$\xi \in \text{Im}(T_e i) = T_e i(\mathfrak{h}) \iff \exp_G(t\xi) \in i(H) \forall t \in \mathbb{R}.$$

1.2 Lie group actions and quotient manifolds

Definition 11 A Lie group action or just action of a Lie group G on a manifold M is a smooth map $\Psi : G \times M \rightarrow M$ that satisfies the usual conditions for a group action. We will denote $\Psi(g, x) = g \cdot x$ if we have a left action, and $\Psi(g, x) = x \cdot g$ for a right action.

It is immediate that each $\Psi_g : M \rightarrow M$ defined by $\Psi_g(x) = \Psi(g, x)$ is a diffeomorphism with inverse $\Psi_{g^{-1}}$.

We remind that an action $\Psi : G \times M \rightarrow M$ is said to be transitive if it has only one orbit and called free if the stabilisator (or isotropy subgroup) G_x of any element $x \in M$ is trivial.

Definition 12 An action $\Psi : G \times M \rightarrow M$ is said to be proper if the map $\hat{\Psi} : G \times M \rightarrow M \times M : (g, x) \mapsto (\Psi(g, x), x)$ is proper.

Proposition 13 *Suppose $F : M \mapsto N$ is a proper continuous map between topological manifolds, then F is closed.*

Proof. See [5], p.47. ■

We now come to one of the most fundamental theorem of smooth manifold theory.

Theorem 14 *Quotient Manifold Theorem*

Suppose a Lie group G acts smoothly, freely, and properly on a smooth manifold M . Then the orbit space M/G is a topological manifold of dimension equal to $\dim(M) - \dim(G)$, and has a unique smooth structure with the property that the quotient map $\pi : M \mapsto M/G$ is a smooth submersion.

Proof. A detailed proof can be found in [5], pp 218-223. The main point of the proof is the existence of so-called adapted charts.

Let $k = \dim(G)$, and $n = \dim(M) - \dim(G)$. We say that a smooth chart (U, φ) on M , with coordinate function $(x, y) = (x^1, \dots, x^k, y^1, \dots, y^n)$ is adapted to the G -action if:

- (i) $\varphi(U)$ is a product open set $U_1 \times U_2 \subset \mathbb{R}^k \times \mathbb{R}^n$, and
- (ii) each orbit intersects U either in the empty set or in a simple slice of the form $\{y^1 = c^1, \dots, y^n = c^n\}$.

It is then shown that every point $p \in M$ admits an adapted coordinate chart centered at p . It is also proven that in this case, every orbit of the action is an embedded submanifold of M .

The key to understanding what it means to be a tangent vector in the quotient is that the $G \cdot p$ component of the chart is moded out, and thus we get the following fundamental identification for the tangent space of M/G at $\pi(p)$:

$$T_{\pi(p)}(M/G) \approx T_p M / T_p(G \cdot p).$$

■

We have noted the proof to this theorem uncovers the fact that all orbits are embedded submanifolds. However, one isn't always confronted to the ideal setting of a free and proper action, and orbits needn't be embedded. However, they are always immersed.

If H is a closed subgroup of a Lie group G , it can be shown that the action by left or right multiplication of H on G is free, proper and smooth. This underlies the following disussion.

For a surjective submersion, local sections show us that a map f out of a

quotient is smooth if and only if it's composition with π , $f \circ \pi$, is smooth. For a general action $\Phi : G \times M \mapsto M$, one can fix an $x \in M$, and define $\Phi_x : G \mapsto M :: g \mapsto g \cdot x$. Consider the right action of G_x on G . G_x is closed, acts freely, properly and smoothly on G , and the map $\tilde{\Phi}_x : G/G_x \rightarrow M :: gG_x \mapsto g \cdot x$ is then a well defined injective smooth map that makes the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{\Phi_x} & M \\ \pi \downarrow & \nearrow \tilde{\Phi}_x & \\ G/G_x & & \end{array}$$

Theorem 15 *If $\Psi : G \times M \mapsto M$ is an action and $x \in M$, then $\tilde{\Psi}_x : G/G_x \mapsto G \cdot x \subset M$ is an injective immersion. If Ψ is proper, the orbit $G \cdot x$ is a closed submanifold of M and $\tilde{\Psi}_x$ is a diffeomorphism.*

Proof. See also [4], p. 265.

We show that $\tilde{\Phi}_x$ is an immersion. Recall from the quotient manifold theorem that for $g \in G$,

$$T_{\pi(g)}(G/G_x) = T_g G / T_g(g \cdot G_x). \quad (1)$$

We treat the case $g = e$, from which the general case follows immediately by a left translation. We show that

$$T_e G_x = \{\xi \in T_e G : T_e \Phi_x \cdot \xi = 0\}.$$

Then, since $T_{\pi(e)} \tilde{\Phi}_x \circ T_e \pi \cdot \xi = T_e \Phi_x$,

$$\begin{aligned} T_{\pi(e)} \tilde{\Phi}_x \cdot [\xi] = 0 & \iff T_e \Phi_x \cdot \xi = 0 \\ & \iff \xi \in \ker T_e \Phi_x \\ & \iff [\xi] = 0, \end{aligned}$$

because of (1).

The inclusion $\subset$ is immediate, we prove the opposite one.

Suppose $\xi \in T_e G$, with $T_e \Phi_x \cdot \xi = 0$. The relation $\Phi_x \circ L_g = \Phi^g \circ \Phi_x$ for all any g in G , x in M , yields by differentiation $T\Phi_x \circ TL_g \cdot \xi = T\Phi^g \circ T\Phi_x \cdot \xi$. Identifying $g = \exp(t\xi)$ in the following calculation, we get

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \Phi_x(\exp(t\xi)) &= T_e \Phi_x(\exp(t\xi)) \circ T(L_{\exp(t\xi)}) \cdot \xi \\ &= T_e \Phi_{\exp(t\xi)}(x) \circ T\Phi_x \cdot \xi \\ &= 0. \end{aligned}$$

By uniqueness of flows, we thus have that $\Phi_x(\exp(t\xi)) = \Phi_x(e) = x \forall t \in \mathbb{R}$. Thus $\exp(t\xi) \in G_x \forall t$, which is equivalent to $\xi \in T_e G_x$, the Lie algebra of G_x .

The last part of the assertion is easily shown. ■

1.3 Infinitesimal generators of an action

Let us be given a smooth action $\Phi : G \times M \rightarrow M$. If $\xi \in T_e G$, then it is immediate to verify that

$$\begin{aligned} \Phi^\xi : \mathbb{R} \times M &\rightarrow M \\ (t, x) &\mapsto \Phi(\exp(t\xi), x) \end{aligned}$$

is an action. In other words, Φ^ξ is a complete flow on M . The infinitesimal generator corresponding to this action is denoted by ξ_M :

$$\xi_M(x) = \left. \frac{d}{dt} \right|_{t=0} \Phi(\exp(t\xi), x).$$

It is crucial to remark, as shown in the previous section and adapting to the language of infinitesimal generators, that

$$T_x(G \cdot x) = \{\xi_M(x) : \xi \in \mathfrak{g}\}.$$

1.3.1 The adjoint action

Definition 16 *Let G be a Lie Group. The action*

$$\begin{aligned} I : G \times G &\rightarrow G \\ (g, h) &\mapsto ghg^{-1} \end{aligned}$$

is called action of G on itself by conjugation.

It is immediate to check that for all $g \in G$, I_g is a homomorphism of Lie groups.

Definition 17 *Let G be a Lie group. The adjoint action to an element g of G is the Lie algebra homomorphism $Ad_g = T_e(L_g \circ R_{g^{-1}})$ induced by I_g . This yields an action*

$$\begin{aligned} Ad : G \times \mathfrak{g} &\rightarrow \mathfrak{g} \\ (g, \xi) &\mapsto T_e(L_g \circ R_{g^{-1}})\xi, \end{aligned}$$

called the adjoint action of G on $T_e G = \mathfrak{g}$.

Definition 18 We denote by ad_ξ the linear map $ad_\xi(\eta) = [\xi, \eta]$.

Proposition 19 For all $\xi \in \mathfrak{g}$,

$$\xi_{\mathfrak{g}} = ad_\xi. \quad (2)$$

Furthermore

$$T_g(Ad.\eta)\xi_g = [T_g R_{g^{-1}}\xi_g, Ad_g\eta], \quad \xi_g \in T_g G, \quad \eta \in \mathfrak{g} \quad (3)$$

where the dot on the right designates the variable.

Proof. Let us denote by θ_t the flow of the left invariant vector field X_ξ on G generated by ξ . Then

$$\begin{aligned} ad_\xi(\eta) &= [\xi, \eta] \\ &= [X_\xi, X_\eta](e) \\ &= (L_{X_\xi} X_\eta)(e) \\ &= \left. \frac{d}{dt} \right|_{t=0} \theta_t^*(X_\eta(\theta_t(e))) \\ &= \left. \frac{d}{dt} \right|_{t=0} [T_{\exp(t\xi)} R_{\exp(-t\xi)}] \cdot X_\eta(\exp(t\xi)) \\ &= \left. \frac{d}{dt} \right|_{t=0} T_{\exp(t\xi)} R_{\exp(-t\xi)} T_e L_{\exp(t\xi)} \eta \\ &= \left. \frac{d}{dt} \right|_{t=0} T_e (L_{\exp(t\xi)} \circ R_{\exp(-t\xi)}) \eta \\ &= \left. \frac{d}{dt} \right|_{t=0} Ad_{\exp(t\xi)} \eta \\ &= \xi_{\mathfrak{g}}(\eta). \end{aligned}$$

We can reformulate this as

$$\begin{aligned} [\xi, \eta] &= \left. \frac{d}{dt} \right|_{t=0} Ad_{\exp(t\xi)} \eta \\ &= T_g(Ad.\eta)\xi. \end{aligned}$$

Thus

$$\begin{aligned} T_g(Ad.\eta)T_e R_g \xi &= \left. \frac{d}{dt} \right|_{t=0} Ad_{R_g(\exp(t\xi))} \eta \\ &= \left. \frac{d}{dt} \right|_{t=0} Ad_{\exp(t\xi)} Ad_g \eta \\ &= [\xi, Ad_g \eta]. \end{aligned}$$

Equivalently,

$$T_g(Ad.\eta)\xi_g = [T_g R_{g^{-1}}\xi_g, Ad_g\eta].$$

■

1.3.2 The coadjoint action

One can dualize the construction of the adjoint, this yields a left action on the dual of the lie algebra, or $\mathfrak{g}^*$.

$$\begin{aligned} Ad^* : G \times \mathfrak{g}^* &\rightarrow \mathfrak{g}^* \\ (g, \alpha) &\mapsto Ad_{g^{-1}}^* \alpha \end{aligned}$$

Remark 20 One has $(Ad^*)_g = (Ad_{g^{-1}})^*$. In the rest of this article, the notation Ad_g^* will mean $(Ad_g)^*$. One doesn't want to invent an new name for an action that is just the dual of another one, but in order to make it left invariant, the inverse should not be forgotten.

Proposition 21 For the Ad^* action, the following holds

$$\xi_{\mathfrak{g}^*} = -ad_{\xi}^* \quad (4)$$

where $(ad_{\xi}^* \alpha)\eta = \alpha(ad_{\xi}(\eta)) = \alpha([\xi, \eta])$, and for all $\xi_g \in T_g G$,

$$T_g(Ad^* \mu)(\xi_g) = Ad_g^*(ad_{T_g R_{g^{-1}}(\xi_g)}^* \mu). \quad (5)$$

Proof. This is proven in the same way as in the adjoint case. ■

The next proposition relates the adjoint action with infinitesimal generators.

Proposition 22 Infinitesimal generators and the adjoint action

Let $\Phi : G \times M \rightarrow M$ be a smooth action. For every $g \in G$ and $\xi, \eta \in T_e G$, we have:

- (i) $(Ad_g \xi)_M = \Phi_{g^{-1}}^* \xi_M$,
- (ii) $[\xi_M, \eta_M] = -[\xi, \eta]_M$.

Proof. See [6] p 269.

Let $x \in M$,

$$\begin{aligned} (Ad_g \xi)_M(x) &= \left. \frac{d}{dt} \right|_{t=0} \Phi(\exp(t Ad_g \xi), x) \\ &= \left. \frac{d}{dt} \right|_{t=0} \Phi(g \exp(t \xi) g^{-1}, x) \\ &= \left. \frac{d}{dt} \right|_{t=0} \Phi_g \circ \Phi(\exp(t \xi), \Phi_{g^{-1}}(x)) \\ &= T_{\Phi_{g^{-1}}(x)} \Phi_g \left. \frac{d}{dt} \right|_{t=0} \Phi(\exp(t \xi), \Phi_{g^{-1}}(x)) \\ &= T_{\Phi_{g^{-1}}(x)} \Phi_g \xi_M(\Phi_{g^{-1}}(x)) \\ &= (\Phi_{g^{-1}}^* \xi_M)(x). \end{aligned}$$

This proves (i). For (ii), plug in $g = \exp(t\xi)$, differentiate and use (2) ■

Let $\tilde{X}_\xi(g) = T_e R_g \xi$ be the right invariant vector field on G generated by ξ . It is straightforward to verify that $\tilde{X}_\xi = \xi_G$, where the underlying action is G acting on itself by left multiplication. Then part (ii) of our last theorem tells us that $[\tilde{X}_\xi, \tilde{X}_\eta] = -\tilde{X}_{[\xi, \eta]}$. We will need this remark in our computation of the symplectic form on coadjoint orbits.

2 The Momentum Map

2.1 Definitions and first properties

Definition 23 A Lie group action $\Phi : G \times P \rightarrow P$ on a symplectic manifold P is called symplectic if $\Phi_g^* \omega = \omega \ \forall g \in G$.

Definition 24 Let $\Phi : G \times P \rightarrow P$ be a symplectic action on a symplectic manifold P . A momentum map for the action Φ is a smooth map

$$J : P \rightarrow \mathfrak{g}^*$$

such that for each $\xi \in \mathfrak{g}$, the associated map

$$\begin{aligned} \hat{J}(\xi) : P &\rightarrow \mathbb{R} \\ x &\mapsto J(x) \cdot \xi \end{aligned}$$

satisfies

$$d(\hat{J}(\xi)) = i_{\xi_P} \omega, \tag{6}$$

where ξ_P is the infinitesimal generator of the action corresponding to the vector ξ .

In other words, J is a momentum map provided for all $\xi \in \mathfrak{g}$,

$$X_{\hat{J}(\xi)} = \xi_P.$$

Remark 25 The pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} &\rightarrow \mathbb{R} \\ (\alpha, \xi) &\mapsto \alpha(\xi) \end{aligned}$$

being smooth, each $\hat{J}(\xi)$ is a smooth function. The map $\hat{J} : \mathfrak{g} \rightarrow C^\infty(P)$ is linear, as is trivially shown. Conversely, having a linear map $\hat{J} : \mathfrak{g} \rightarrow C^\infty(P)$ satisfying the condition of the definition defines a momentum map J by setting $J(x)(\xi) = \hat{J}(\xi)(x)$. J is then smooth because any map into $\mathfrak{g}^*$ is smooth if and only if its pairing with any $\xi \in \mathfrak{g}$ is, which is the case here by definition.

If we are given a symplectic action such that each ξ_P is globally hamiltonian, then there exists a momentum map. Indeed, if $\xi_1, \dots, \xi_k$ is a basis for the lie algebra $\mathfrak{g}$ and $J_1, \dots, J_k$ are the hamiltonians for $(\xi_1)_P, \dots, (\xi_k)_P$, we can define $\hat{J}(\xi_i) = J_i$ and extend by linearity. Condition (6) will be satisfied because both d and i are linear.

If the action has two momentum maps J and J' , and if P is connected, $0 = d(\hat{J}(\xi) - \hat{J}'(\xi))$ thus $\hat{J}(\xi) - \hat{J}'(\xi)$ is locally constant and thus constant for all $\xi \in \mathfrak{g}$, which implies the existence of $\mu \in \mathfrak{g}^*$ such that $J(p) - J'(p) = \mu \forall p \in P$.

Theorem 26 [6] p.277 Conservation of Momentum

Let Φ be a symplectic action of G on a symplectic manifold (P, ω) with momentum map J . Suppose $H : P \rightarrow \mathbb{R}$ is invariant under the action of Φ (i.e. $H(x) = H(\Phi_g(x)) \forall x \in P, g \in G$), then J is an integral for X_H (i.e. if F_t is the flow of X_H then $J(F_t(x)) = J(x)$ for all x and t where F_t is defined).

Proof. Since H is invariant, we have $H(\Phi_{\exp(t\xi)}x) = H(x) \forall \xi \in \mathfrak{g}, \forall t \in \mathbb{R}$. Differentiating at $t = 0$,

$$\begin{aligned} 0 &= dH(x) \cdot \xi_P(x) \\ &= \mathcal{L}_{X_{\hat{J}(\xi)}} H \\ &= \{H, \hat{J}(\xi)\}, \\ &= -X_H \hat{J}(\xi), \end{aligned}$$

which is equivalent to our statement. ■

Let's now explicitly construct momentum maps for certain categories of symplectic manifolds.

Definition 27 A momentum map J is said to be Ad^* -equivariant if the actions are compatible with J , i.e. $J(\Phi_g(x)) = Ad_{g^{-1}}^* J(x) \forall x, \forall g$. The diagram

$$\begin{array}{ccc} P & \xrightarrow{\phi_g} & P \\ J \downarrow & & \downarrow J \\ \mathfrak{g}^* & \xrightarrow{Ad_{g^{-1}}^*} & \mathfrak{g}^* \end{array} \quad (7)$$

commutes.

Theorem 28 Let Φ be a symplectic action of a Lie group G on a symplectic manifold (P, ω) . Assume $\omega = -d\theta$ and that the action leaves θ invariant, that is, $\Phi_g^* \theta = \theta$ for all $g \in G$. Then $J : P \rightarrow \mathfrak{g}^*$ defined by

$$J(x) \cdot \xi = (i_{\xi_P} \theta)(x)$$

defines an Ad^* -equivariant momentum map for the action.

Proof. Since the action leaves θ invariant, we have

$$0 = \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp(t\xi)}^* \theta = L_{\xi_P} \theta.$$

Using Cartan's magic formula, we get

$$d(i_{\xi_P} \theta) + i_{\xi_P} d\theta = 0, \text{ i.e. } d(i_{\xi_P} \theta) = i_{\xi_P} \omega,$$

showing $\hat{J}(\xi) = i_{\xi_P} \theta$ satisfies the condition for being a momentum map.

Now for Ad^* -equivariance, we must show that

$$\begin{aligned} J(\Phi_g(x) \cdot \xi) &= Ad_{g^{-1}}^* J(x) \cdot \xi \\ \iff \hat{J}(\xi)(\Phi_g(x)) &= \hat{J}(Ad_{g^{-1}} \xi)(x) \\ \iff i_{\xi_P} \theta(\Phi_g(x)) &= (i_{(Ad_{g^{-1}} \xi)_P} \theta)(x), \end{aligned}$$

but the last equality is true because $(Ad_{g^{-1}} \xi)_P = \Phi_g^* \xi_P$, so

$$\begin{aligned} &(i_{(Ad_{g^{-1}} \xi)_P} \theta)(x) \\ &= (i_{\Phi_g^* \xi_P} \Phi_g^* \theta)(x) \\ &= (\Phi_g^* (i_{\xi_P} \theta))(x) \\ &= (i_{\xi_P} \theta)(\Phi_g(x)), \end{aligned}$$

using invariance of θ in the second equality. This completes our proof. ■

A very convenient way of constructing symplectic actions is by pulling back an action to its cotangent bundle. The key for specializing the previous theorem to this case is the following theorem.

Theorem 29 Cotangent Lift Theorem

*Given two manifolds S and Q , a diffeomorphism $\varphi : T^*S \rightarrow T^*Q$ preserves the canonical one-forms θ_Q on T^*Q and θ_S on T^*S , respectively, if and only if $\varphi = T^*f$ for some diffeomorphism $f : Q \rightarrow S$.*

Proof. A proof can be found in [4], pp 170-172. Notice that since d commutes with the pullback operation, φ is then a symplectic map for the canonical symplectic structures on the cotangent bundles. ■

If we are given an action $\Phi : G \times Q \rightarrow Q$, by fixing a $g \in G$, we can lift the induced diffeomorphism $\Phi_g : x \mapsto g \cdot x$, to $\Phi_g^* : T^*Q \rightarrow T^*Q$, which is

symplectic. If we do this for every $g \in G$, and patch them together, we get a symplectic G -action on T^*Q :

$$\begin{aligned} \Phi^{T^*} \quad T^*Q &\rightarrow T^*Q \\ (g, \alpha) &\mapsto \Phi_{g^{-1}}^* \alpha. \end{aligned}$$

We now have all the tools to specialize Theorem 29.

Theorem 30 *Let Φ be an action of G on Q and let Φ^{T^*} be the corresponding lifted action on $P = T^*Q$. Then this action is symplectic and has an Ad^* -equivariant momentum map $J : P \rightarrow \mathfrak{g}^*$ defined by*

$$\hat{J}(\xi)(\alpha_q) = \alpha_q \cdot \xi_Q(q).$$

Proof. The projection map on the basepoint, $\tau_Q^* : T^*Q \rightarrow Q$, is equivariant by construction, for all $g \in G$,

$$\tau_Q^* \circ \Phi_g^{T^*} = \Phi_g \circ \tau_Q.$$

Replacing g by $\exp(t\xi)$ and differentiating at $t = 0$, we get

$$T\tau_Q^* \circ \xi_P = \xi_P \circ \tau_Q^*.$$

We now apply Theorem 28 and use the definition of the canonical one-form to verify our formula:

$$\begin{aligned} J(\alpha_q)(\xi) &= (i_{\xi_P} \theta)(\alpha_q) \\ &= \left\langle \alpha_q, T\tau_Q^* \circ \xi_P \right\rangle \\ &= \left\langle \alpha_q, \xi_P \circ \tau_Q^*(\alpha_q) \right\rangle \\ &= \left\langle \alpha_q, \xi_Q(q) \right\rangle. \end{aligned}$$

■

2.2 Momentum maps as a generalization of hamiltonian functions

Let $H \in C^\infty(T^*\mathbb{R}^n) = C^\infty(\mathbb{R}^{2n})$, together with its canonical symplectic form $\omega = \sum dq_i \wedge dp^i$. Associate to H it's hamiltonian vector field X_H and corresponding symplectic flow F . Assume F is complete, it is then a symplectic action $F : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Let's denote the lie algebra of $\mathbb{R}$ by $\mathcal{R} = T_0\mathbb{R}$ and identify it with $\mathbb{R}$ itself in the usual way. We denote elements of $\mathcal{R}$ by $\tilde{1}, \tilde{2}, \tilde{3}, \dots$

The key to our discussion is that the infinitesimal generator of F corresponding to $\tilde{1}$ is X_H . Indeed,

$$X_H(p) = \frac{d}{dt} \Big|_{t=0} F(t, p) = \frac{d}{dt} \Big|_{t=0} F(\exp(t\tilde{1}, p) = (\tilde{1}_{\mathbb{R}^{2n}})(p),$$

since under the usual identification of $\mathbb{R}$ with $\mathcal{R}$, the exponential map is just the identity. In order to have a momentum map J , we can define its associated comoment map by setting $\hat{J}(\tilde{1}) = H$ and extending by linearity: $\hat{J}(\tilde{k}) = kH$, so $J(x) \cdot \tilde{k} = (kH)(x)$. Canonically identifying $\mathbb{R}$ with its dual $\mathbb{R}^*$, we have that $J(x) = H(x)$.

Ad^* -invariance of J is immediate, because first $\mathbb{R}$ is a commutative Lie group, and so the Ad and Ad^* actions are trivial, and secondly H is invariant under the flow of its associated hamiltonian vector field X_H .

2.3 Linear and angular momentum as momentum maps

Linear momentum can be seen as the momentum map for the action of $\mathbb{R}^n$ on itself by left translation. Indeed, let $\Psi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n : (s, q) \rightarrow s + q$, then $\xi_{\mathbb{R}^n}(q) = \xi \ \forall q \in \mathbb{R}^n$. Theorem 30 tells us that $\tilde{J}(\xi) \cdot (p, q) = p \cdot \xi$ on $T^*\mathbb{R}^n$, and so $J(q, p) = p$, the usual linear momentum.

The discussion for angular momentum is more involved, because we need to consider the tangent bundle of $\mathbb{R}^n$, and adapt the theory to this case.

For this, we remember that for a Riemannian manifold Q , there is a bundle isomorphism between its tangent and cotangent bundles, namely $g^b : TQ \rightarrow T^*Q$, $g^b(X_q)Y_q = g(X_q, Y_q)$; its inverse is denoted by $g^\#$. We define the canonical one-form on TQ by $\Theta := (g^b)^*\theta_0$, where θ_0 is the canonical 1-form on the cotangent bundle. It is straightforward to show that $\Omega := -d\Theta = (g^b)^*\omega$ is a symplectic form on TQ .

Lemma 31 *If $f : Q_1 \rightarrow Q_2$ is an isometry of riemannian manifolds, then $Tf : TQ_1 \rightarrow TQ_2$ preserves Θ .*

Proof. We shall prove that

$$Tf = g_2^\# \circ (T^*f)^{-1} \circ g_1^b.$$

Each of these maps preserves the canonical one-form, so their composition does too, which completes the proof. Note that these maps are thus also

symplectic.

For $Y_{f(q)} \in T_{f(q)}Q_2$, $X_q \in T_qQ_1$, we have that

$$\begin{aligned}
(g_2^b \circ Tf)(X_q)(Y_{f(q)}) &= g_2^b(Tf \cdot X_q)(Y_{f(q)}) \\
&= \langle Tf \cdot X_q, Y_{f(q)} \rangle_2 \\
&= \langle Tf \cdot X_q, Tf \circ Tf^{-1} \cdot Y_{f(q)} \rangle_2 \\
&= \langle X_q, Tf^{-1}Y_{f(q)} \rangle_1 \\
&= g_1^b(X_q)(T(f^{-1}) \cdot Y_{f(q)}) \\
&= (T^*(f^{-1}) \circ g_1^b(X_q))(Y_{f(q)}),
\end{aligned}$$

which was to be shown. ■

The adaptation of Theorem 28 to this case, which is proved in the same way as Theorem 30, is as follows:

Proposition 32 *Let G be a Lie group acting isometrically on a Riemannian manifold Q . The tangent lift of this action, which is symplectic by Lemma 31, has an Ad^* -equivariant moment map given by*

$$\hat{J}(\xi)(v_q) = \langle v_q, \xi_Q(q) \rangle.$$

We now examine the lie algebra $\mathfrak{so}(3)$ of $SO(3)$.

Consider the map

$$\begin{aligned}
\Psi : GL(n, \mathbb{R}) &\rightarrow S(n, \mathbb{R}) \\
A &\mapsto A^T A,
\end{aligned}$$

where $S(n, \mathbb{R})$ denotes the $n(n+1)/2$ -linear subgroup of symmetric $n \times n$ matrices of $M(n, \mathbb{R})$. We have $O(n) = \Psi^{-1}(I_n)$. We take global coordinates on $\mathbb{R}^{n^2}$ and show I_n is a regular value of Ψ .

$GL(n, \mathbb{R})$ is open in $M(n, \mathbb{R})$ and so $TGL(n, \mathbb{R}) = GL(n, \mathbb{R}) \times M(n, \mathbb{R})$. For $A \in GL(n, \mathbb{R})$, $B \in M(n, \mathbb{R})$, taking $\gamma(t) = A + tB$, we have

$$\begin{aligned}
\Psi_* B &= (\Psi \circ \gamma)'(0) \\
&= \left. \frac{d}{dt} \right|_{t=0} (A + tB)^T (A + tB) \\
&= B^T A + A^T B.
\end{aligned}$$

For $C \in S(n, \mathbb{R})$, by choosing $B = \frac{1}{2}AC$, we get $\Psi_* B = C$, so Ψ is a submersion and $\Psi^{-1}(I_n) = O(n)$ is an embedded submanifold of dimension $n^2 - n(n+1)/2 = n(n-1)/2$. So $\mathfrak{o}(n) = \ker(T_{Id}\Psi) = \{A \in M(n, \mathbb{R}) : T_{Id}\Psi \cdot A = A^T + A = Id\}$, the space of $n \times n$ skew-symmetric matrices. It can be shown

that $O(n)$ has two connected components, $SO(n)$ being the one containing the identity. Thus $\mathfrak{so}(n) = \mathfrak{o}(n)$.

One can identify $\mathfrak{so}(n)$ (with its commutator bracket for matrices Lie algebra) with $\mathbb{R}^3$ seen as a Lie algebra with Lie bracket $[x, y] = x \times y$ (remember this is defined intrinsically!). Define

$$\hat{\cdot} : \mathbb{R}^3 \mapsto \mathfrak{so}(n) : v = (v_1, v_2, v_3) \mapsto \hat{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

We have $\hat{v}w = v \times w$. Thus

$$\begin{aligned} (\hat{u}\hat{v} - \hat{v}\hat{u})w &= \hat{u}(v \times w) - \hat{v}(u \times w) \\ &= u \times (v \times w) - v \times (u \times w) \\ &= (u \times v) \times w \\ &= \widehat{(u \times v)}w. \end{aligned}$$

So $\hat{u}\hat{v} - \hat{v}\hat{u} = \widehat{u \times v}$.

It can be shown reasonably easily, see for example [4], pp 290-291, that $\exp(t\hat{w})$ is a rotation about w of angle $t\|w\|$, where $w \in \mathbb{R}^3$. Thus if we consider the natural action of $SO(3)$ on $\mathbb{R}^3$, it is straightforward that for $\xi \in \mathbb{R}^3$ one gets $\hat{\xi}_{\mathbb{R}^3}(x) = \xi \times x$ (a direct calculation also shows that the adjoint action $SO(3)$ on its Lie algebra seen as $\mathbb{R}^3$ is just the natural action of evaluation).

We now at last have all the tools necessary to realize the angular momentum $\xi \times x$ as a momentum map.

Let $\Xi : SO(3) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (A, x) \mapsto Ax$ be the usual action, then its lift $T\Xi$ is a symplectic action by Lemma 31. Hence, denoting the inverse of $w \mapsto \hat{w}$ by $B \mapsto \tilde{B}$, we get

$$\begin{aligned} \hat{J}(B)(q, v) &= \langle \tilde{B} \times q, v \rangle \\ &= \det(\tilde{B}, q, v) \\ &= \det(q, v, \tilde{B}) \\ &= \langle q \times v, \tilde{B} \rangle. \end{aligned}$$

If we identify $\mathfrak{so}(3, \mathbb{R})$ with $\mathbb{R}^3$ on the one hand, and $\mathbb{R}^3$ with its dual on the other, we get the angular momentum as a momentum map.

3 The Marsden-Weinstein-Meyer Quotient theorem

3.1 Symplectic reduction and reduction of dynamics

Theorem 33 [6] pp. 299-300 *Marsden-Weinstein-Meyer Reduction Theorem*

Let (P, ω) be a symplectic manifold on which the Lie group G acts symplectically and let $J : P \rightarrow \mathfrak{g}^*$ be an Ad^* -equivariant momentum map for this action. Assume $\mu \in \mathfrak{g}^*$ is a regular value of J and that the isotropy group G_μ under the Ad^* action on $\mathfrak{g}^*$ acts freely and properly on $J^{-1}(\mu)$. Then $P_\mu = J^{-1}(\mu)/G_\mu$ has a unique symplectic form ω_μ with the property

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega$$

where $\pi_\mu : J^{-1}(\mu) \rightarrow P_\mu$ is the canonical projection and $i_\mu : J^{-1}(\mu) \rightarrow P$ is the inclusion.

For this we need the following.

Lemma 34 Let $p \in J^{-1}(\mu)$, then

- (i) $T_p(G_\mu \cdot p) = T_p(G \cdot p) \cap T_p(J^{-1}(\mu))$, and
- (ii) $T_p(J^{-1}(\mu))$ is the ω -orthogonal complement of $T_p(G \cdot p)$.

Proof. We recall that

$$\begin{aligned} T_p(G \cdot p) &= \{\xi_P(p) : \xi \in \mathfrak{g}\}, \\ T_p(G_\mu \cdot p) &= \{\xi_P(p) : \xi \in \mathfrak{g}_\mu\}, \end{aligned} \tag{8}$$

where $\mathfrak{g}_\mu$ is the Lie algebra of G_μ . Thus our statement is equivalent to the condition

$$\xi_P(p) \in T_p(J^{-1}(\mu)) \iff \xi \in \mathfrak{g}_\mu.$$

We know J is Ad^* -equivariant, thus $T_p J(\xi_P(p)) = \xi_{\mathfrak{g}^*}(\mu)$, so we have

$$\xi_P(p) \in T_p(J^{-1}(\mu)) = \ker(T_p J) \iff \xi_{\mathfrak{g}^*}(\mu) = 0.$$

The condition $\xi_{\mathfrak{g}^*}(\mu) = 0$ is equivalent to the integral curve of the induced left action $(t, \nu) \mapsto Ad_{\exp(-t\xi)}\nu$ on $\mathfrak{g}^*$ by ξ being constant, by uniqueness of integral curves. Stated another way, this is equivalent to μ being a fixed point of the induced action. However, $\exp(t\xi) \in G_\mu, \forall t \in \mathbb{R} \iff \xi \in \mathfrak{g}_\mu$, which completes part (i).

For part (ii), we use that

$$\omega(\xi_P(p), v_p) = d(\hat{J}(\xi))_p \cdot v_p = \langle T_p J \cdot v_p, \xi \rangle$$

where $v_p \in T_p P$, $\xi \in \mathfrak{g}$. So

$$\begin{aligned}
v \in T_p(J^{-1}(\mu)) = \ker(T_p J) &\iff T_p J \cdot v_p = 0 \\
&\iff (T_p J \cdot v_p)\xi = 0 \quad \forall \xi \in \mathfrak{g} \\
&\iff \omega(\xi_P(p), v_p) = 0 \quad \forall \xi \in \mathfrak{g} \\
&\iff v_p \in (T_p(G \cdot p))^\omega,
\end{aligned}$$

the last equality being true because of (8). ■

Proof. (Marsden-Weinstein Theorem)

We denote $[v_p] := T\pi_\mu \cdot v_p$, and recall that $T_{\pi(p)}P_\mu \approx T_p(J^{-1}(\mu))/T_p(G_\mu \cdot p)$, as quotient of a vector space by a subspace. We have

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega \iff \omega_\mu([v], [w]) = \omega(v, w) \quad \forall v, w \in T_p(J^{-1}(\mu)).$$

Well-definedness follows immediatly from part (ii) of Lemma (34). Since π_μ and $T\pi_\mu$ are surjective, our form ω_μ is uniquely defined, should it satisfy $\pi_\mu^* \omega_\mu = i_\mu^* \omega$.

For smoothness of ω_μ , we recall that a surjective submersion admits a smooth local section σ at any point of its image, and verify immediatly that locally, $\omega_\mu = \sigma^* \omega$.

Closedness can be checked similarly with local sections, denoted by σ :

$$\omega_\mu = \sigma^* \omega \Rightarrow d(\omega_\mu) = d(\sigma^* \omega) = \omega_\mu = \sigma^*(d\omega) = 0.$$

It remains only to be shown that ω_μ is nondegenerate. This is true since

$$\begin{aligned}
\omega_\mu([v], [w]) \quad \forall w \in T_p(J^{-1}(\mu)) &\Rightarrow \omega(v, w) = 0 \quad \forall w \in T_p(J^{-1}(\mu)) \\
&\Rightarrow v \in T_p(G_\mu \cdot p) \\
&\Rightarrow [v] = 0.
\end{aligned}$$

■

Theorem 35 *Reduction of Dynamics*

Under the assumptions of the Marsden-Weinstein reduction Theorem, let $H : P \rightarrow \mathbb{R}$ be invariant under the action of G . Then we have:

- (i) *The flow F of X_H leaves $J^{-1}(\mu)$ invariant,*
- (ii) *it commutes with the action of G_μ on $J^{-1}(\mu)$, and thus induces canonically a flow H_t on P_μ satisfying $\pi_\mu \circ F_t = H_t \circ \pi_\mu$.*
- (iii) *The flow H_t is a Hamiltonian flow on P_μ with Hamiltonian H_μ defined by $H_\mu \circ \pi_\mu = H \circ i_\mu$. We call it the **reduced Hamiltonian**.*

Proof. Let's denote our action by Φ . First, remark that since this action is symplectic, X_H is Φ_g -related to itself: for all g in G ,

$$\Phi_g^* X_H = X_{H \circ \Phi_g} = X_H.$$

Thus, the flow F of X_H commutes with the action of G on P (for more details, see for example [5], Lemma 18.4).

By Theorem 26, we know that J is constant on the flow F of X_H , thus we can restrict F to $J^{-1}(\mu)$. Let's now fix a value $t \in \mathbb{R}$, and take $F_t : J^{-1}(\mu) \rightarrow J^{-1}(\mu)$ down to the quotient by defining

$$H_t \circ \pi_\mu(x) := \pi_\mu \circ F_t(x).$$

H_t is well-defined since F_t commutes with the action of G . It is smooth because $\pi_\mu \circ F_t$ is. The following commutative diagram will be useful to keep in mind.

$$\begin{array}{ccccc} J^{-1}(\mu) & \xrightarrow{F_t} & J^{-1}(\mu) & \xrightarrow{i_\mu} & P \xrightarrow{H} \mathbb{R} \\ \pi_\mu \downarrow & & \pi_\mu \downarrow & \nearrow H_\mu & \\ J^{-1}(\mu)/G_\mu & \xrightarrow{H_t} & J^{-1}(\mu)/G_\mu & & \end{array} \quad (9)$$

We have

$$\pi_\mu^* H_t^* \omega_\mu = F_t^* \pi_\mu^* \omega_\mu = F_t^* i_\mu^* \omega = i_\mu^* \omega = \pi_\mu^* \omega_\mu,$$

the third equality as a result of the flow F being symplectic. Taking local sections, we conclude that $H_t^* \omega_\mu = \omega_\mu$, so H_t is symplectic. Let us show that the infinitesimal generator Y of H_t is globally Hamiltonian. For this, define $H_\mu : P_\mu \rightarrow R$ by $H_\mu \circ \pi_\mu = H \circ i_\mu$. This is well defined by G -invariance of H . Since H_t and F_t are π_μ -related for all t , we have that $Y(\pi_\mu(p)) = T\pi_\mu \cdot X_H(p)$. Let $[v] = T\pi_\mu \cdot v \in TP_\mu$. Then

$$\begin{aligned} dH_\mu \cdot v &= dH_\mu(T\pi_\mu \cdot v) \\ &= (T\pi_\mu \cdot v)H_\mu \\ &= v(H_\mu \circ \pi_\mu) \\ &= v(H \circ i_\mu) \\ &= (Ti_\mu \cdot v)H \\ &= dH(Ti_\mu \cdot v) \\ &= (i_\mu^* dH) \cdot v \\ &= (i_\mu^* \omega)(X_H, v). \\ &= (\pi_\mu^* \omega_\mu)(X_H, v) \\ &= \omega_\mu(Y, [v]). \end{aligned}$$

We have shown that Y has a global hamiltonian function H_μ , which was what we wanted. ■

3.2 Coadjoint orbits as symplectic manifolds

We use the Marsden-Weinstein-Meyer reduction procedure to build a symplectic structure on the coadjoint orbits of the Ad^* -action induced by any Lie group G . This procedure and the explicit symplectic form in terms of infinitesimal generators that we will devise is called the *Kirillov-Kostant-Souriau theorem*.

We are given a Lie group G , together with its left action $\Lambda : G \times G \rightarrow G : (g, h) \rightarrow L_g h$. We lift this action to the cotangent bundle T^*G and obtain a symplectic action

$$\begin{aligned} \Lambda^{T^*} : G \times T^*G &\rightarrow T^*G \\ (g, \alpha_g) &\mapsto T^*L_{g^{-1}}\alpha_g. \end{aligned}$$

The momentum map $J : T^*G \rightarrow \mathfrak{g}^*$ is then given by Theorem 30, for $\xi \in \mathfrak{g}$:

$$J(\alpha_g) \cdot \xi = \alpha_g(\xi_G(g)) = \alpha_g \cdot T_e R_g(\xi).$$

Indeed

$$\xi_G(g) = \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi) \cdot h = \left. \frac{d}{dt} \right|_{t=0} R_h \exp(t\xi) = T R_h(\xi)$$

since $T_{0_g} \exp \cong Id_{\mathfrak{g}}$.

Thus

$$J(\alpha_g) = T_e^* R_g \cdot \alpha_g = \alpha_g \circ T_e R_g.$$

Let's show that every value $\mu \in \mathfrak{g}^*$ is a regular value for J . For this consider the diffeomorphism $\Phi : T^*G \rightarrow G \times \mathfrak{g}^* : (g, \alpha_g) \mapsto (g, T_e^* R_g \cdot \alpha_g)$. Then the following diagram commutes, that is, $\Phi \circ \pi_2 = J$, where π_2 is projection onto the second factor, from which our assertion follows immediately.

$$\begin{array}{ccc} T^*G & \xrightarrow{J} & \mathfrak{g}^* \\ \Phi \downarrow & \nearrow \pi_2 & \\ G \times \mathfrak{g}^* & & \end{array}$$

Now $J^{-1}(\mu) = \{(g, \alpha_g) \in T^*G : \mu = T_e^* R_g \cdot \alpha_g\}$. So

$$\begin{aligned} (g, \alpha_g) \in J^{-1}(\mu) &\iff \alpha_g \circ T_e R_g = \mu \\ &\iff T_e^* R_g \cdot \alpha_g = \mu \\ &\iff \alpha_g = T_g^* R_{g^{-1}} \cdot \mu. \end{aligned}$$

This is nothing else than the image of the right-invariant covector field α_μ induced by the value μ at the identity:

$$\begin{aligned} \alpha_\mu : G &\rightarrow T^*G \\ g &\mapsto (g, \alpha_g = T_g^* R_{g^{-1}} \cdot \mu). \end{aligned}$$

Let us see who G_μ is.

$$\begin{aligned} g \in G_\mu &\iff \text{Ad}_\mu^* \mu = \mu \\ &\iff T_e^*(L_g \circ R_{g^{-1}}) \mu = \mu \\ &\iff T_e^*(R_{g^{-1}} \circ L_g) \mu = \mu \\ &\iff \alpha_\mu(g) = T_g^* L_{g^{-1}} \mu = T_g^* L_{g^{-1}} \alpha_\mu(e) \\ &\iff L_{g^{-1}}^* \alpha_\mu = \alpha_\mu \\ &\iff L_g^* \alpha_\mu = \alpha_\mu, \end{aligned}$$

so G_μ is the set of all elements that leave the form α_μ invariant by left translation.

Although this is not correlated to the explicit identification of G_μ we have just made, the action Λ^{T^*} acts on points in T^*G by shifting the base point to the left. So, by projecting $J^{-1}(\mu)$ to G , we can view the action of G_μ on $J^{-1}(\mu)$ as simply the left action of G_μ on G ! This action is free and proper, so we can form the quotient manifold G/G_μ and use the machinery we developed in the first chapter to get the following diffeomorphisms

$$\begin{aligned} J^{-1}(\mu)/G_\mu &\cong G/_L G_\mu \cong G/_R G_\mu \cong G \cdot \mu \subseteq \mathfrak{g}^* \\ \pi_\mu(g, \alpha_\mu(g)) &\mapsto [g] \mapsto [g^{-1}] \mapsto \text{Ad}_g^* \mu, \end{aligned}$$

where $/_L$ and $/_R$ denote the quotients by the left and right action of G_μ on G , respectively.

Using the Marsden-Weinstein-Meyer quotient theorem and pushing forward the symplectic form to $G \cdot \mu$, we have shown that the coadjoint orbit at a point μ is a symplectic manifold.

We now move to the task of explicitly computing the symplectic form ω_μ on the coadjoint orbit of μ . Let us define

$$\begin{aligned} \zeta : J^{-1}(\mu) &\rightarrow G \cdot \mu \\ (g, \alpha_\mu(g)) &\mapsto \text{Ad}_g^* \mu, \end{aligned}$$

so that by the Marsden-Weinstein-Meyer theorem and our construction, $\zeta^* \omega_\mu = i_\mu^* \omega$, where ω is the canonical two form on T^*G .

We follow [6], p. 303. This will be done in several steps.

- $J^{-1}(\mu)$ is the image of G by the right-invariant one-form α_μ . Thus $T_{\alpha_\mu(g)}J^{-1}(\mu) = T\alpha_\mu(T_gG) = T\alpha_\mu TR_g(\mathfrak{g})$.
- We do the following computation:

$$\begin{aligned}
& i_\mu^* \omega(T\alpha_\mu TR_g \xi, T\alpha_\mu TR_g \eta) \\
&= \alpha_\mu^* \omega(TR_g \xi, TR_g \eta) \\
&= -d\alpha_\mu(TR_g \xi, TR_g \eta) \\
&= -d\alpha_\mu(\tilde{X}_\xi, \tilde{X}_\eta)(g) \\
&= -\{\tilde{X}_\xi(\alpha_\mu(\tilde{X}_\eta)) - \tilde{X}_\eta(\alpha_\mu(\tilde{X}_\xi)) - \alpha_\mu([\tilde{X}_\xi, \tilde{X}_\eta])\}(g) \\
&= \alpha_\mu([\tilde{X}_\xi, \tilde{X}_\eta])(g) \\
&= -\mu([\xi, \eta]),
\end{aligned}$$

where $\tilde{X}_\eta$ denotes the right-invariant vector field extending η . The fourth equality holds because

$$\alpha_\mu(\tilde{X}_\eta) = (T^*R_{g^{-1}}\mu)(TR_g\eta) = \mu(\eta)$$

is constant. The last equality follows from $[\tilde{X}_\xi, \tilde{X}_\eta] = -\tilde{X}_{[\xi, \eta]}$.

- We have

$$\begin{aligned}
& (\zeta^* \omega_\mu)(T\alpha_\mu TR_g \xi, T\alpha_\mu TR_g \eta) \\
&= \omega_\mu(Ad^*\mu)(T_{\alpha_\mu(g)}\zeta \cdot T_g\alpha_\mu \cdot T_eR_g\xi, T_{\alpha_\mu(g)}\zeta \cdot T_g\alpha_\mu \cdot T_eR_g\eta) \\
&= \omega_\mu(T_g(Ad^*\mu) \cdot T_eR_g\xi, T_g(Ad^*\mu) \cdot T_eR_g\eta),
\end{aligned}$$

since $\zeta(\alpha_\mu(g)) = Ad_g^*\mu$.

- Now comes the more subtle part, where one has to be careful how to view our objects. One has

$$\begin{aligned}
T_g(Ad^*\mu) \cdot T_eR_g\xi &= Ad_g^*(ad_{T_gR_{g^{-1}}T_eR_g\xi}^*\mu) \\
&= Ad_g^*(ad_{\xi^*}^*\mu) \\
&= -Ad_g^*(\xi_{\mathfrak{g}^*}(\mu)) \\
&\cong -(T_\mu Ad_g^*)(\xi_{\mathfrak{g}^*}(\mu)) \\
&= -((Ad_{g^{-1}}^*)^*\xi_{\mathfrak{g}^*})(Ad_g^*\mu) \\
&= -((Ad^*)_g)^*\xi_{\mathfrak{g}^*}(Ad_g^*\mu) \\
&= -(Ad_{g^{-1}}\xi)_{\mathfrak{g}^*}(Ad_g^*\mu)
\end{aligned}$$

The first equality follows from (5), the second is just taking care of the subscripts, the third by (5). The last follows from (22).

Putting these four steps together, we get

$$\omega_\mu(Ad_g^*\mu)((Ad_{g^{-1}}\xi)_{\mathfrak{g}^*}(Ad_g^*\mu), Ad_{g^{-1}}\eta)_{\mathfrak{g}^*}(Ad_g^*\mu)) = -\mu \cdot [\xi, \eta].$$

Making the change of variables

$$\begin{aligned} Ad_{g^{-1}}\xi' &= \xi \\ Ad_{g^{-1}}\eta' &= \eta \\ \nu &= Ad_g^*\mu, \end{aligned}$$

and remembering that for a fixed g , Ad_g is a Lie algebra homomorphism, we finally get the elegant formula

$$\omega_\mu(v)(\xi_{\mathfrak{g}^*}(v), \eta_{\mathfrak{g}^*}(v)) = -v \cdot [\xi, \eta].$$

It is straightforward to verify (by direct calculation or by using the left action of G on T^*G) that under the action of $Ad_{g^{-1}}^*$ on $G \cdot \mu$, the symplectic form ω_μ is preserved. Thus our orbit is a homogeneous Hamiltonian G -space.

3.3 The complex projective spaces as real symplectic manifolds

The complex projective space $\mathbb{CP}^{n-1}$ is by definition the topological quotient of $\mathbb{C}^n \setminus \{0\}$ by $\mathbb{C}^*$ by the action

$$\begin{aligned} M : \quad \mathbb{C}^* \times \mathbb{C}^n \setminus \{0\} &\rightarrow \mathbb{C}^n \setminus \{0\} \\ (\lambda, (a_1, \dots, a_n)) &\mapsto (\lambda a_1, \dots, \lambda a_n). \end{aligned}$$

For an element $A = (a_1, \dots, a_n)$ such that $\|A\| = 1$, every $\lambda = \exp(i\theta)$ will satisfy $\|\lambda A\| = 1$. We can thus get $\mathbb{CP}^{n-1}$ through the action

$$\begin{aligned} m : \quad S^1 \times S^{2n-1} &\rightarrow S^{2n-1} \\ (\lambda, (a_1, \dots, a_n)) &\mapsto (\lambda a_1, \dots, \lambda a_n). \end{aligned}$$

To put a symplectic form on $\mathbb{CP}^{n-1} = S^{2n-1}/S^1$, the idea is to identify S^{2n-1} as the integral curve (or orbit) of a complete periodic hamiltonian vector field (whose flow is then symplectic).

This can be done with the so called "harmonic oscillator hamiltonian" $H \in C^\infty(T^*\mathbb{R}^n) : H(q^i, p_i) = \frac{1}{2} \sum_i ((q^i)^2 + (p_i)^2)$, with the canonical symplectic form $\omega = -d\theta = \sum_i dq^i \wedge dp_i$. See also [4], p302. We get

$$X_H(q^i, p_i) = \sum_{i=1}^n (p_i \frac{\partial}{\partial q^i} - q^i \frac{\partial}{\partial p_i})$$

which integrates to the complete symplectic action

$$\begin{aligned} F : \mathbb{R} \times T^*\mathbb{R}^n &\rightarrow T^*\mathbb{R}^n \\ (t, (\mathbf{q}, \mathbf{p})) &\mapsto (\mathbf{q} \cos t + \mathbf{p} \sin t, \mathbf{p} \cos t - \mathbf{q} \sin t). \end{aligned}$$

Each orbit is 2π -periodic and so we can see this action as a symplectic action of S^1 on $T^*\mathbb{R}^n$. S^{2n-1} appears as $H^{-1}(\frac{1}{2})$, and is evidently a regular value! S^1 is compact, so the action is proper. It is also free. The quotient theorem allows us to conclude.

4 An Introduction to Brownian Motion

This section is based primarily on [1] and [2], and should provide a quick but nevertheless rigorous introduction to brownian motion. Most proofs are taken from these sources, reorganized and completed.

4.1 First things first

4.1.1 Reminder of measure theory

We go through this chapter in an informal manner. Omitted proofs can be found in any introductory text in probability.

Definition 36 *Let f be a measurable function between a probability space $(\Omega, \mathcal{F}, P)$ and a measurable space $(\Omega^*, \mathcal{F}^*)$. We define the image measure of P by f , or pushforward of P by f , or law of f , the probability measure f_*P on $(\Omega^*, \mathcal{F}^*)$ defined by $f_*P(F) = P(f^{-1}(F))$ for all F in $\mathcal{F}^*$.*

Proposition 37 *With the same notations as in definition (36). Let g be a measurable function from $(\Omega^*, \mathcal{F}^*)$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then the following holds*

$$g \in \mathcal{L}^1(\Omega^*, \mathcal{F}^*, f_*P) \iff g \circ f \in \mathcal{L}^1(\Omega, \mathcal{F}, P),$$

in which case we have

$$\int_{\Omega^*} g \, d(f_*P) = \int_{\Omega} (g \circ f) \, dP.$$

Proposition 38 Dynkin's Lemma

Let E be a set and Π be a π -system in E (subset of $\mathcal{P}(E)$ that is stable by finite intersection) such that $E \in \Pi$. Let Λ be a λ -system (subset of $\mathcal{P}(E)$ that is stable by complement and countable disjoint union) containing Π . Then $\sigma(\Pi) \subseteq \Lambda$ (the smallest sigma algebra containing Π is contained in Λ , or the intersection of all sigma algebras containing Π is contained in Λ).

This proposition is a jack in the box in probability theory because it pops up absolutely everywhere. It allows us to focus on a special type of generating sets for a sigma algebra instead of the whole sigma algebra. Let us illustrate this by proving two key lemmas.

Lemma 39 *Let μ, ν be two finite measures on a measurable space $(\Omega, \mathcal{F})$ that agree on a π -system Π that generates $\mathcal{F}$ and contains $\mathcal{F}$ as a subset. Then $\mu(A) = \nu(A)$ for all A in $\mathcal{F}$.*

Proof. Just verify that the set of all sets in $\mathcal{F}$ on which μ and ν agree is a λ -system, which is immediate. ■

An application of this Lemma is uniqueness of the measure giving the volume of hypercubes on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, or Lebesgue measure on borelians.

This Lemma will also give us uniqueness of the Wiener measure, as we shall see later on.

Lemma 40 *Let $\Pi_1, \dots, \Pi_n$ be π -systems on a probability space $(\Omega, \mathcal{F}, P)$, with $\mathcal{F} \in \Pi_i, \forall i \in \{1, \dots, n\}$. Suppose*

$$P(F_1 \cap \dots \cap F_n) = \prod_{i=1}^n P(F_i) \quad \forall F_1 \in \Pi_1, \dots, F_n \in \Pi_n, \quad (10)$$

then $\sigma(\Pi_1), \dots, \sigma(\Pi_n)$ are independant sigma algebras.

Proof. Fix $F_2 \in \Pi_2, \dots, F_n \in \Pi_n$, and verify that the set of all sets F_1 in $\mathcal{F}$ such that (10) is true is a λ -system containing Π_1 . Once this is done for every $F_2 \in \Pi_2, \dots, F_n \in \Pi_n$, move on to the second factor, with $F_1 \in \sigma(\Pi_1), F_3 \in \Pi_3, \dots, F_n \in \Pi_n$ and apply the same procedure until the last factor is reached. ■

4.1.2 Characteristic functions

Definition 41 *Let X be an $\mathbb{R}^d$ -valued random variable on a probability space $(\Omega, \mathcal{F}, P)$. The characteristic function of X , or Fourier transform of the measure X_*P , is defined by*

$$\Phi(\xi) = \mathbb{E}[\exp(i \langle \xi, X \rangle)] = \int_{\mathbb{R}^d} \exp(i \langle \xi, x \rangle) d(X_*P), \quad \forall \xi \in \mathbb{R}^d.$$

Notice the integrand is bounded by 1, thus the integral is always defined, since we are on a probability space. This function is uniformly continuous, C^n differentiable if $n \leq [p]$, with $X \in \mathcal{L}^p$.

The admirable property of Fourier transforms of measures on $\mathbb{R}^d$ is stated in the following theorem.

Theorem 42 *The characteristic function of an $\mathbb{R}^d$ -valued random variable X characterizes the law of this random variable. In other words, the Fourier transform, as defined on the set of all measures on $\mathbb{R}^d$, is injective.*

4.1.3 Gaussian Vectors

Definition 43 A random variable $X : (\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$ is called gaussian of mean m and standard value σ if

$$X_*P = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right)dx,$$

where dx is the Lebesgue measure on $\mathbb{R}$. Equivalently, in terms of characteristic functions, $\Phi_X(\xi) = \exp(-\frac{1}{2}\xi^2\sigma^2 + im\xi)$, $\xi \in \mathbb{R}$. We then write $X \rightsquigarrow \mathcal{N}(m, \sigma^2)$.

A key observation is noticing that a sum of independant gaussian random variables is still a gaussian random variable. This can be shown by means of the Fourier transform. If $X_1 \rightsquigarrow \mathcal{N}(m_1, \sigma_1^2)$, $X_2 \rightsquigarrow \mathcal{N}(m_2, \sigma_2^2)$, then

$$\begin{aligned} & \mathbb{E}[\exp(i(X_1 + X_2)\xi)] \\ &= \mathbb{E}[\exp(iX_1\xi)] \cdot \mathbb{E}[\exp(iX_2\xi)] \\ &= \exp(-\frac{1}{2}\xi^2(\sigma_1^2 + \sigma_2^2) + i(m_1 + m_2)\xi) \end{aligned}$$

and so $X_1 + X_2 \rightsquigarrow \mathcal{N}(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$.

Definition 44 Let $\mathcal{C}$ be a symmetric semi-positive $d \times d$ matrix with real coefficients. A random variable $X = (X_1, \dots, X_d) : (\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}^d$ is called centred gaussian vector of covariance matrix $\mathcal{C}$ if

$$\Phi_X(\xi) = \exp\left(-\frac{1}{2}\xi^T \mathcal{C} \xi\right), \quad \xi \in \mathbb{R}^d.$$

We then write $X \rightsquigarrow \mathcal{N}(0, \mathcal{C})$.

Theorem 45 Gaussian Random Vectors

Let $\mathcal{C}$ be a symmetric semi-positive $d \times d$ matrix with real coefficients, then there exists a centred gaussian vector X of covariance matrix $\mathcal{C}$. The components of X are then gaussian variables and their covariance matrix is given by $\mathcal{C}$, thus its name. Conversely, if $X = (X_1, \dots, X_d)$ is composed by d gaussian variables with covariance matrix $\mathcal{C}$, then X is gaussian of covariance matrix $\mathcal{C}$.

In other terms, using injectivity of the Fourier transform, the covariance matrix of Gaussian random variables completely determines their joint probability law. In particular, gaussian random variables are independant if and only if their covariance matrix is diagonal. A key to proving this theorem

is the principle axis theorem: there exists an orthogonal change of basis A such that $X = (X_1, \dots, X_d) = A \cdot Y$, with Y a gaussian vector made of independant gaussian variables (i.e. the covariance matrix is diagonal). In particular a vector valued random variable is gaussian if and only if any linear combination of its components is gaussian.

Here is the last key ingredient in our discussion of gaussian vectors.

Theorem 46 Vectorial central limit theorem

Let $(X_n)_{n \geq 1}$ be a sequence of identically distributed independant square integrable $\mathbb{R}^d$ -valued random variables. Let K be the covariance matrix of the components of one of these variables. Then

$$\frac{1}{\sqrt{n}}(X_1 + \dots + X_n) \xrightarrow[n \rightarrow \infty]{(law)} \mathcal{N}(0, K).$$

4.2 Brownian motion as a limit of a scaled random walk

Definition 47 One calls brownian motion (starting at 0, in dimension d) a family of $\mathbb{R}^d$ -valued random variables $(B_t)_{t \in \mathbb{R}_+}$ such that:

- (P1) one has $B_0 = 0$ a.s. Furthermore, for any choice of integer $p \geq 1$ and real numbers $0 = t_0 < t_1 < \dots < t_p$, the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_p} - B_{t_{p-1}}$ are independant, and for any $j \in \{1, \dots, p\}$, $B_{t_j} - B_{t_{j-1}}$ is a centered gaussian vector of covariance matrix $(t_j - t_{j-1})Id$.
- (P2) For every $\omega \in \Omega$, the function $t \mapsto B_t(\omega)$ is continuous.

Remark 48 To show that a brownian motion actually exists is all but evident. A proof can be found in [1], pp 222-225; a second proof can be found in [2], chapters 1 and 2, along with a detailed study of gaussian variables and measures. It should be noted that both proofs involve an isometric embedding of some hilbert space into a gaussian space (or closed subspace of some $L^2(\Omega, \mathcal{A}, P)$ that is made up of centred gaussian variables). The key here is that for Gaussian random variables, zero covariance (or zero scalar product in this L^2 space) is equivalent to (and not only implied by!) independance. The proof for $d = 1$ immediatly extends to any $d \in \mathbb{N}$ by taking d independant copies of a brownian motion in one dimension.

We now show how to obtain a brownian motion as a limit of a scaled random walk. For this, let's consider a random walk $(S_n)_{n \in \mathbb{N}}$ on $\mathbb{Z}^d$ starting at zero, i.e. we are given independant identically distributed random variables $(Y_i)_{i \in \mathbb{N}^*}$, and set $S_0 = 0$, $S_n = \sum_{i=1}^n Y_i$, for $n \geq 1$. Some hypotheses will be necessary. First, we would like to use the power of the central limit theorem,

and so will assume that $Y_i \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$, i.e. $\sum_{k \in \mathbb{Z}^d} \|k\|^2 d(Y_{i*}P) < \infty$. We will assume our variables are centered: $E(Y_i) = (0, \dots, 0)$. Finally, we want our random walk to reflect isotropy, i.e. the same behavior in any direction of space. This is translated mathematically by saying that the covariance function of any two components of Y_i is given by: $Cov(Y_i^\alpha, Y_i^\beta) = \sigma^2 \delta_{\alpha\beta}$, $\alpha, \beta \in \{1, \dots, d\}$, respectively $\sum_{k \in \mathbb{Z}^d} k_\alpha k_\beta d(Y_{i*}P) = \sigma^2 \delta_{\alpha\beta}$.

Proposition 49 *For any $p \geq 1$, and any choice of real numbers $0 = t_0 < t_1 < \dots < t_p$, one has*

$$(S_{t_1}^{(n)}, S_{t_2}^{(n)}, \dots, S_{t_p}^{(n)}) \xrightarrow[n \rightarrow \infty]{(law)} (U_1, U_2, \dots, U_p),$$

where

$$S_t^{(n)} := \frac{1}{\sqrt{n}} S_{[nt]}$$

is a change of scale of the function $k \mapsto S_k$, and the limit law is characterized by:

- (i) the random variables $U_1, U_2 - U_1, \dots, U_p - U_{p-1}$ are independent;
- (ii) for any $j \in \{1, \dots, p\}$, $U_j - U_{j-1}$ is a centered gaussian vector of covariance matrix $\sigma^2(t_j - t_{j-1})Id$, where by convention $U_0 = 0$.

Proof. We only need to show that for any $\xi_1, \dots, \xi_p \in \mathbb{R}^d$,

$$\mathbb{E}[\exp(i \sum_{j=1}^p \xi_j \cdot S_{t_j}^{(n)})] \xrightarrow[n \rightarrow \infty]{} \mathbb{E}[\exp(i \sum_{j=1}^p \xi_j \cdot U_j)],$$

since convergence in law is equivalent to convergence of the Fourier transform (a result of Lévy, c.f. [1], pp 133-134). This is equivalent to

$$\begin{aligned} & \mathbb{E}[\exp(i \sum_{j=1}^p \eta_j \cdot (S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}))] \\ & \xrightarrow[n \rightarrow \infty]{} \mathbb{E}[\exp(i \sum_{j=1}^p \eta_j \cdot (U_j - U_{j-1}))] \quad \forall \eta_1, \dots, \eta_p \in \mathbb{R}^d. \end{aligned}$$

However, using independance and the Fourier transform of Gaussian variables,

$$\begin{aligned} & \mathbb{E}[\exp(i \sum_{j=1}^p \eta_j \cdot (U_j - U_{j-1}))] \\ & = \prod_{j=1}^p \mathbb{E}[\exp(i \eta_j \cdot (U_j - U_{j-1}))] \\ & = \exp(-\sum_{j=1}^p \frac{\sigma^2 |\eta_j|^2 (t_j - t_{j-1})}{2}). \end{aligned}$$

On the one hand, we have

$$S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)} = \frac{1}{\sqrt{n}} \sum_{k=\lfloor nt_{j-1} \rfloor + 1}^{\lfloor nt_j \rfloor} Y_k,$$

which tells us that the random variables $S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}$ are independant for $1 \leq j \leq p$. On the other hand, for a fixed value of j ,

$$\begin{aligned} & S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)} \\ & \stackrel{(law)}{=} \frac{1}{\sqrt{n}} S_{\lfloor nt_j \rfloor - \lfloor nt_{j-1} \rfloor} \\ & = \frac{\sqrt{\lfloor nt_j \rfloor - \lfloor nt_{j-1} \rfloor}}{\sqrt{n}} \frac{1}{\sqrt{\lfloor nt_j \rfloor - \lfloor nt_{j-1} \rfloor}} S_{\lfloor nt_j \rfloor - \lfloor nt_{j-1} \rfloor}. \end{aligned}$$

Using the condition of isotropy and the central limit theorem for vector valued random variables, we conclude that this variable converges in law to a normal centered variable $\mathcal{N}$ of covariance matrix $\sigma \cdot Id$.

As a consequence, for every fixed j ,

$$\begin{aligned} & \mathbb{E}[\exp(i\eta_j \cdot (S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}))] \\ & \xrightarrow{n \rightarrow \infty} \mathbb{E}[\exp(i\sqrt{t_j - t_{j-1}}\eta_j \cdot \mathcal{N})] \\ & = \exp(-\frac{\sigma^2 |\eta_j|^2 (t_j - t_{j-1})}{2}). \end{aligned}$$

Independance of the random variables $S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)}$, $1 \leq j \leq p$ now allows us to conclude. ■

4.3 The Wiener measure

We would like to change our point of view, and see an element ω of our probability space $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$ as a path on its own right. At the same time, we would like to have a way to identify all brownian motions. The solution to this problem can be given by putting an adequate sigma algebra on $C(\mathbb{R}_+, \mathbb{R}^d)$, the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, and endowing it with a pushforward measure, or Wiener measure. Furthermore, this space is very convenient for time-shifting of paths, such as we will explain in the setting of the weak and strong Markov properties.

Definition 50 *Let $\mathcal{C}$ be the smallest sigma algebra on $C(\mathbb{R}_+, \mathbb{R}^d)$ that makes all coordinate projection functions π_t , $t \in \mathbb{R}_+$ measurable.*

Remark 51 *It is trivial to verify that a function f from a measurable space into $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{C})$ is measurable if and only if $\pi_i \circ f$ is measurable for every i in $\mathbb{R}_+$.*

Remark 52 *It can be shown, for example in [1], page 226, that $\mathcal{C}$ coincides with the borel sigma algebra (or sigma algebra generated by a topology) of the topology of uniform convergence on every compact for $C(\mathbb{R}_+, \mathbb{R}^d)$.*

Definition 53 *Let $(B_t)_{t \in \mathbb{R}_+}$ be a d -dimensional brownian motion starting at zero, defined on a space $(\Omega, \mathcal{F}, P)$. The Wiener measure in dimension d is then defined to be the pushforward measure $\mathbb{P}_0 = \Phi_* P$ of the map*

$$\begin{aligned} \Phi : \Omega &\longrightarrow C(\mathbb{R}_+, \mathbb{R}^n) \\ \omega &\longmapsto (B_t(\omega))_{t \in \mathbb{R}_+}. \end{aligned}$$

A few remarks should immediatly be made.

First, Φ is measurable, because it's component functions $\pi_t \circ \Phi, t \geq 0$ are precisely the measurable functions $B_t, t \geq 0$.

Secondly, the following calculation guarantees us that $\mathbb{P}_0$ is uniquely defined, whatever the choice of brownian motion we are starting with. For this, let $A_0, A_1, \dots, A_p$ be borel sets of $\mathbb{R}^d$, then

$$\begin{aligned} &\mathbb{P}_0(\{\omega \in C(\mathbb{R}_+, \mathbb{R}^n) : w(t_0) \in A_0, w(t_1) \in A_1, \dots, w(t_p) \in A_p\}) \\ &= P(B_{t_0} \in A_0, B_{t_1} \in A_1, \dots, B_{t_p} \in A_p) \\ &= 1_{A_0}(0) \int_{A_1 \times \dots \times A_p} dy_1 \dots dy_p p_{t_1}(y_1) p_{t_2-t_1}(y_2 - y_1) \dots p_{t_p-t_{p-1}}(y_p - y_{p-1}), \end{aligned}$$

where p_σ denotes the density of a gaussian variable $X \rightsquigarrow \mathcal{N}(0, \sigma \cdot Id)$.

We conclude once again with Dynkin's Lemma, all sets of the form $\{\omega \in C(\mathbb{R}_+, \mathbb{R}^n) : w(t_0) \in A_0, w(t_1) \in A_1, \dots, w(t_p) \in A_p\}$ forming a π -system generating $\mathcal{C}$.

The interpretation of this construction should be the following: the Wiener measure is a probability on the set of continuous paths $C(\mathbb{R}_+, \mathbb{R}^n)$ such that under this measure the projection functions constitute a brownian motion. We call this the *canonical brownian motion*.

It should be mentioned that the exact same construction can be made for a brownian motion starting at another point x than the identity, in which case we call the corresponding Wiener measure $\mathbb{P}_x$.

4.4 The weak Markov Property and Blumenthal's zero-one law

We consider a brownian motion B in $\mathbb{R}^d$. We shall note, for every $s \geq 0$, $\mathcal{F}_s = \sigma(B_r, 0 \leq r \leq s)$, $\mathcal{F}_\infty = \sigma(B_r, r \leq \infty)$.

Proposition 54 *Some properties of brownian motion*

(i) if φ is a vectorial isometry of $\mathbb{R}^d$, then $(\varphi(B_t))_{t \in \mathbb{R}_+}$ is also a brownian motion

(ii) for each $\gamma > 0$, the process $B_t^\gamma = \frac{1}{\gamma} B_{\gamma^2 t}$ is also a brownian motion (scale invariance);

(iii) for each $s > 0$, the process $B_t^{(s)} := B_{s+t} - B_s$ is a brownian motion independant of $\mathcal{F}_s$ (weak Markov property)

Proof. For both (i) and (ii), we shall use injectivity of the fourier transform of measures on $\mathbb{R}^n$

Let's prove (i): denote by $\varphi(x) = Ax$ an orthogonal transformation on $\mathbb{R}^n$, i.e. $A^T A = I$. If X is a Gaussian vector, then so is AX . We trivially have $\varphi(B_0) = 0$ a.s. and the new paths are continuous. Now let $0 < t_1 < t_2 < \dots < t_p$. Since $B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_p} - B_{t_{p-1}}$ are gaussian and independant, the same is true for $\varphi(B_{t_1}), \varphi(B_{t_2} - B_{t_1}), \dots, \varphi(B_{t_p} - B_{t_{p-1}})$. Now we compute for $s < t$,

$$\begin{aligned} & \mathbb{E}[\exp(i \langle \xi, \varphi(B_t - B_s) \rangle)] \\ &= E[\exp(i \langle A^T \xi, B_t - B_s \rangle)] \\ &= \exp(-\frac{1}{2} \xi^T A(t-s) I A^T \xi) \\ &= \exp(-\frac{1}{2} \xi^T (t-s) I \xi), \end{aligned}$$

which proves that $\varphi(B_t - B_s)$ has law $\mathcal{N}(0, (t_i - t_{i-1})I)$.

For (ii), $s < t$, the same computations lead us to

$$\begin{aligned} & \mathbb{E}[\exp(i \langle \xi, B_t^\gamma - B_s^\gamma \rangle)] \\ &= \mathbb{E}[\exp(i \langle \xi, \frac{1}{\gamma} B_{\gamma^2 t} - B_{\gamma^2 s} \rangle)] \\ &= \exp(-\frac{1}{2} (\frac{\xi}{\gamma})^T (\gamma^2 t - \gamma^2 s) I \frac{\xi}{\gamma}) \\ &= \exp(-\frac{1}{2} \xi^T (t-s) I \xi), \end{aligned}$$

which is what we wanted. The last property is an easy application of Dynkin's lemma. ■

Another elegant way of stating the weak Markov property is to define a shift operator

$$\begin{aligned} \theta_t : C(\mathbb{R}_+, \mathbb{R}^n) &\longrightarrow C(\mathbb{R}_+, \mathbb{R}^n) \\ w &\longmapsto w(t + \cdot), \end{aligned}$$

to verify this map is measurable and the pushforward satisfies $\theta_t^* \mathbb{P}_0 = \mathbb{P}_0$. As we shall see later on, the exact same procedure goes through when we replace a deterministic time t with a stopping time T . This is then called the strong Markov Property, and contains the weak Markov property as a special case for a constant stopping time.

Theorem 55 *Blumenthal's Zero-One Law*

Let

$$\mathcal{F}_{0+} = \bigcap_{t>0} \mathcal{F}_t,$$

then $\mathcal{F}_{0+}$ is trivial in the sense that $A \in \mathcal{F}_{0+} \implies P(A) = 0$ or $P(A) = 1$.

Proof. Let $A \in \mathcal{F}_{0+}$ and let $t_1, \dots, t_p > 0$, $p \in \mathbb{N}^*$. For $\epsilon > 0$ small enough, Markov's weak property implies that $(B_{t_1} - B_\epsilon, \dots, B_{t_p} - B_\epsilon)$ is independent of $\mathcal{F}_\epsilon$, and thus of $\mathcal{F}_{0+}$. As a consequence, for every continuous bounded function f on $(\mathbb{R}^d)^p$,

$$\mathbb{E}[\mathbf{1}_A f(B_{t_1} - B_\epsilon, \dots, B_{t_p} - B_\epsilon)] = P(A) \mathbb{E}[f(B_{t_1} - B_\epsilon, \dots, B_{t_p} - B_\epsilon)].$$

By letting ϵ go to zero, we get by dominated convergence

$$\mathbb{E}[\mathbf{1}_A f(B_{t_1}, \dots, B_{t_p})] = P(A) \mathbb{E}[f(B_{t_1}, \dots, B_{t_p})],$$

and thus $(B_{t_1}, \dots, B_{t_p})$ is independent of $\mathcal{F}_{0+}$. As a consequence of Dynkin's Lemma, $\bigcup \{\sigma(B_{t_1}, \dots, B_{t_p}), t_1, \dots, t_p > 0, p \in \mathbb{N}^*\} \cup \mathcal{F}_0$ being a π -system generating $\mathcal{F}_\infty$, we find that $\mathcal{F}_\infty$ is independent of $\mathcal{F}_{0+}$. In particular, $\mathcal{F}_{0+}$ is independent of itself. But this can only happen if $\mathcal{F}_{0+}$ is trivial. ■

Remark 56 *It should be pointed out that when we applied Dynkin's Lemma, we implicitly used the fact that $\mathcal{F}_0$ is trivial, and thus independent of any sigma algebra, in particular of $\mathcal{F}_{0+}$.*

Corollary 57 *Assume $d = 1$, then almost surely, for every $\epsilon > 0$*

$$\sup_{0 \leq s \leq \epsilon} B_s > 0; \inf_{0 \leq s \leq \epsilon} B_s < 0. \quad (11)$$

(Carefully look at the order of logical quantifiers here, it is NOT the other way round, we have a better assertion here). Define, for every $a > 0$, $T_a = \inf\{t \geq 0 : B_t = a\}$. One sets $\inf \emptyset = \infty$. Then

$$a.s. \forall a \in \mathbb{R}, T_a < \infty.$$

As a consequence, almost surely,

$$\limsup_{t \rightarrow \infty} B_t = +\infty; \liminf_{t \rightarrow \infty} B_t = -\infty.$$

Remark 58 *The supremum in the first inequalities of the corollary are taken over an uncountable set. However, one can restrict ourselves to the values of s in $[0, \epsilon] \cap \mathbb{Q}$, the paths being continuous.*

Proof. The first assertion is a direct consequence of Blumenthal's zero-one law. The second is a consequence of the first assertion using the property of invariance of scale. The last property follows from the second by remarking the following: any continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is surjective if and only if $\limsup_{t \rightarrow \infty} f_t = +\infty$ & $\liminf_{t \rightarrow \infty} f_t = -\infty$, by a trivial compactness argument.

Let's prove Theorem 11. Let (ϵ_p) be a sequence of strictly positive reals decreasing to 0. Let

$$A := \bigcap_p \left\{ \sup_{0 \leq s \leq \epsilon} B_s > 0 \right\}$$

We prove that $P(A) = 1$. It is clear that A is $\mathcal{F}_{0+}$ -measurable and that

$$P(A) = \lim_{p \rightarrow \infty} \downarrow P\left(\sup_{0 \leq s \leq \epsilon} B_s > 0\right).$$

But

$$P\left(\sup_{0 \leq s \leq \epsilon} B_s > 0\right) \geq P(B_{\epsilon_p} > 0) = \frac{1}{2},$$

since B_{ϵ_p} has a gaussian $\mathcal{N}(0, \epsilon_p)$ distribution. Blumenthal's zero-one law concludes the proof. The assertion for the infimum is obtained by replacing B by $-B$.

For the second assertion, we write

$$1 = P\left(\sup_{0 \leq s \leq 1} B_s > 0\right) = \lim_{\delta \downarrow 0} \uparrow P\left(\sup_{0 \leq s \leq 1} B_s > \delta\right),$$

and use the property of invariance of scale of brownian motion

$$P\left(\sup_{0 \leq s \leq 1} B_s > \delta\right) = P\left(\sup_{0 \leq s \leq \frac{1}{\delta^2}} B_s^\delta > 1\right) = P\left(\sup_{0 \leq s \leq \frac{1}{\delta^2}} B_s > 1\right),$$

the last equality by uniqueness of the law of brownian motion again. Letting δ go to zero, we find that

$$P(\sup_{s \geq 0} B_s > 1) = 1.$$

Let $A > 0$, we use uniqueness of the law of brownian motion and a change of scale to find that

$$1 = P(\sup_{s \geq 0} B_s^A > 1) = P(\sup_{t \geq 0} B_t > A),$$

since $\{\sup_{s \geq 0} \frac{B_{A^2 s}}{A} > 1\} = \{\sup_{t \geq 0} B_t > A\}$. The assertion for the infimum is once again obtained by replacing B by $-B$. ■

4.5 The strong Markov property and the reflexion principle

We would like to extend the weak Markov property from deterministic stopping times to random stopping times. We keep the same notations as in the previous section.

Definition 59 *A random variable T taking values in $[0, \infty]$ is called a stopping time if $\forall t \geq 0, \{T \leq t\} \in \mathcal{F}_t$.*

Definition 60 *Let T be a stopping time. The sigma algebra of events anterior to T , is*

$$\mathcal{F}_T := \{A \in \mathcal{F}_\infty : \forall t \geq 0, A \cap \{T \leq t\} \in \mathcal{F}_t\}.$$

Lemma 61 *T and $1_{\{T < \infty\}} B_T$ are $\mathcal{F}_T$ - measurable.*

Proof. The assertion for T is trivial. The second assertion uses continuity of trajectories of a brownian motion: notice that

$$1_{\{T < \infty\}} B_T = \lim_{n \rightarrow \infty} \sum_{i=0}^{\infty} 1_{\{i2^{-n} \leq T < (i+1)2^{-n}\}} B_{i2^{-n}}, \quad (12)$$

so that we will be able to conclude if we can show that $\forall s \geq 0, 1_{\{s \leq T\}} B_s$ is $\mathcal{F}_T$ - measurable. Indeed, since

$$\forall i : 1_{\{i2^{-n} \leq T < (i+1)2^{-n}\}} B_{i2^{-n}} = 1_{\{i2^{-n} \leq T < (i+1)2^{-n}\}} 1_{\{i2^{-n} \leq T\}} B_{i2^{-n}},$$

we just use the fact that a product of measurable functions is measurable and conclude by stability of measurable functions under $\liminf$ or $\limsup$ operations.

Now let $A \in \mathcal{B}(\mathbb{R}), 0 \notin A$. We know that for any function $f : (\Omega, \mathcal{F}) \rightarrow \Omega^*$, the set of sets $\{B \subset \Omega : f^{-1}(B) \in \mathcal{F}\}$ is a sigma algebra, as can readily be checked. However $\sigma\{A \in \mathcal{B}(\mathbb{R}), 0 \notin A\} = \mathcal{B}(\mathbb{R})$, so we are done if we can prove that $(1_{\{s \leq T\}} B_s)^{-1}(A) \cap \{T \leq t\} \in \mathcal{F}_t \forall A, 0 \notin A, \forall t \geq 0$. For $s \leq t$, this is clear because then $1_{\{s \leq T\}} B_s$ is $\mathcal{F}_t$ - measurable. For $s > t$, we notice that $(1_{\{s \leq T\}} B_s)^{-1}(A) \cap \{T \leq t\} = B_s^{-1}(A) \cap \{s \leq T\} \cap \{T \leq t\} = \emptyset \in \mathcal{F}_t$.

Theorem 62 Markov's Strong Property

Let T be a stopping time such that $P(T > 0) > 0$. Then conditionnally to $\{T > 0\}$ (i.e. the probability measure is rescaled to this set), the stochastic process $B^{(T)}$ defined by

$$B_t^{(T)} = B_{T+t} - B_T$$

is a brownian motion independant of $\mathcal{F}_T$, where one has to define $B^{(T)}$ on the set $\{\omega : T = \infty\}$, for example by setting $B_t^{(T)}(w) = 0 \forall t$ on that set.

Proof. The proof can be found in [1], on page 230; although quite technical, it requires little more than the previous Lemma and Dynkin's Lemma again. ■

We now come to a fundamental property of brownian motion in one dimension.

Theorem 63 *The Reflexion Principle*

One supposes $d = 1$. For every $t > 0$, denote $S_t = \sup_{s \leq t} B_s$. Then, if $a \geq 0$, and $b \leq a$, one has

$$P(S_t \geq a, B_t \leq b) = P(B_t \geq 2a - b).$$

In Particular, S_t has the same law as $|B_t|$.

Remark 64 *One should have in mind the image of a path that reaches height "a" on the y-axis at some time $s \in [0, t]$, the reflexion principle then tells us it has the same probability of travelling "a - b" distance in one direction or the other in the remaining time. Although intuitively evident, we need most of the technology built so far to prove this fact rigorously.*

Proof. We apply Markov's strong property to the stopping time

$$T_a = \inf\{t \geq 0, B_t = a\}.$$

We already know by Corollary 57 that $T_a < \infty$ almost surely. Now

$$\begin{aligned} & P(S_t \geq a, B_t \leq b) \\ &= P(T_a \leq t, B_t \leq b) \\ &= P(T_a \leq t, B_{t-T_a}^{(T_a)} \leq b - a), \end{aligned}$$

since $B_{t-T_a}^{(T_a)} = B_t - B_{T_a} = B_t - a$. We note $B' := B^{(T_a)}$, so that by Theorem 62, B' is a brownian motion that is independant of $\mathcal{F}_{T_a}$, thus of T_a . Since B' has the same law as $-B'$, the couple (T_a, B') has the same law as $(T_a, -B')$. Denote

$$H = \{(s, w) \in \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}) : s \leq t \text{ and } w(t-s) \leq b-a\}.$$

Then our previous probability is

$$\begin{aligned} & P((T_a, B') \in H) \\ &= P((T_a, -B') \in H) \\ &= P(T_a \leq t, -B_{t-T_a}^{(T_a)} \leq b-a) \\ &= P(T_a \leq t, B_t \geq 2a-b) \\ &= P(B_t \geq 2a-b) \end{aligned}$$

where the last equality come from the fact that $2a - b \geq a$, thus $\{B_t \geq 2a - b\} \subseteq \{T_a \leq t\}$.

For the second part of the assertion, note that

$$\begin{aligned} P(S_t \geq a) &= P(S_t \geq a, B_t \geq a) + P(S_t \geq a, B_t \leq a) \\ &= 2P(B_t \geq a) \\ &= P(|B_t| \geq a) \end{aligned}$$

where the second equality follows from the reflexion principle for the second term and from the fact that $\{S_t \geq a, B_t \geq a\} = \{B_t \geq a\}$ for the first. ■

Remark 65 *One still should have to prove the set H is measurable. This is quite technical, if someone knows of a more easy proof, he or she should please tell me about it!*

First, as proven in [1] on page 226, the sigma algebra $\mathcal{C}$ on $C(\mathbb{R}_+, \mathbb{R}^n)$ is the borelian sigma algebra of the topology of uniform convergence on any compact on $C(\mathbb{R}_+, \mathbb{R}^n)$. Two things should be said: first, on $C(\mathbb{R}_+, \mathbb{R}^n)$, the uniform convergence on any compact topology coincides with the compact open topology, and secondly, it is metrizable and separable. Once again, see [1], page 226. Now we know that in the event of seperable metrizable spaces E and F , $\mathcal{B}(E \otimes F) = \mathcal{B}(E) \otimes \mathcal{B}(F)$, which means that the sigma algebra generated by the product topology is the product sigma algebra of the sigma algebras generated by the seperate topologies. We now conclude by remembering that the evaluation map $e_t : \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^n) \rightarrow \mathbb{R}_+ :: (s, w) \rightarrow w(t - s)$ is continuous with the compact open topology on $C(\mathbb{R}_+, \mathbb{R}^n)$ and remark that $H = e_t^{-1}(\mathbb{R}_+)$.

4.6 Lévy's characterization of Brownian motion

For those who are familiar with stochastic integration, here is a characterization theorem for brownian motion, the proof can be found in [2], pp 75-76.

Theorem 66 *Lévy's characterization of brownian motion*

Let $X = (X^1, \dots, X^d)$ be a continuous, $(\mathcal{F}_t)$ -adapted processes starting at zero. Equivalent are:

(i) X is an $(\mathcal{F}_t)$ -brownian motion

(ii) The processes $X^1, \dots, X^d$ are $(\mathcal{F}_t)$ -continuous local martingales and furthermore $\langle X^i, X^j \rangle = \delta_{ij}t$

In particular, an $(\mathcal{F}_t)$ -continuous local martingale M starting at zero is an $(\mathcal{F}_t)$ -brownian motion if and only if $\langle M, M \rangle_t = t \forall t \geq 0$.

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